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UNITARY EXTENSIONS OF TWO-PARAMETER LOCAL SEMIGROUPS OF ISOMETRIC OPERATORS AND THE KREIN EXTENSION THEOREM

Ramón Bruzual

A notion of two-parameter local semigroups of isometric operators in Hilbert space is discussed. It is shown that under certain conditions such a semigroup can be extended to a strongly continuous two-parameter group of unitary operators in a larger Hilbert space. As an application a simple proof of the Eskin bidimensional version of the Krein extension theorem is given.

1.-INTRODUCTION

In several problems of the Analysis, and in Quantum Physics a notion of one parameter local semigroups of operator appears, and in some basic instances the local semigroup extends to a group of unitary operators. In particular local semigroups appear in some problems on Fourier representation of positive definite functions of a real variable, where the unitary extensions of the semigroup provide solutions of the problem. To such problems belong also the classical theorem of Krein which asserts that every continuous positive definite function defined in an interval $I \subset \mathbb{R}$ can be extended to a continuous positive definite function in $\mathbb{R}$. The theory of one parameter local semigroup of isometries in Pontrjagyn spaces was started by Grossman and Langer [12] who proved the existence of unitary extensions of such semigroups and derived from this result a generalization of Krein's theorem for $\varkappa$-indefinite functions.

In the case of Hilbert space such a theory was developed independently by the author in [04] for the more general case of local semigroups of contractions, giving several applications to Generalized Toeplitz kernels as well as a very simple proof of the Krein extension theorem. The results of [04] were developed further in [05] for the Krein-Schwartz case.

In [06] a theory of local semigroups of operators, and in particular of isometries in Pontrjagyn spaces was given, deriving as a corollary the theorem of Grossman-Langer-Krein and a similar result for $\varkappa$-indefinite generalized Toeplitz kernels.

It is known that the Krein theorem may fail in the two-dimensional case, i.e. not every continuous positive definite function defined in $I_1 x I_2 \subset \mathbb{R}^2$ extends to a positive definite function in $\mathbb{R}^2$. However Devinatz [07] proved that such an extension exists if the positive definite function satisfies some additional conditions, which were later relaxed by Eskin [09] (see section 3). On the other hand the notion of two parameter local semigroups of isometries can be defined in a natural way and a pair of infinitesimal generators can be attached to it. Peláez ([22]) gave a necessary and sufficient condition for a two parameter local semigroup in a Hilbert space to extend to a two parameter unitary group (see theorem 2.4 below), and he combined this theorem with some results of Devinatz [07] to obtain a new proof of the above mentioned theorem of Devinatz. However theorem 2.4 alone wouldn't provide a simple proof of Devinatz result, since that theorem reduces the problem of the existence of unitary extensions of the semigroup to an equally difficult classical problem for the associated pair of generators. This last problem has been studied by several authors (see [03]), and Koranyi [16] gave a particular simple sufficient condition for its solution.

In the present paper we discuss in greater detail the problem of unitary extensions for two parameter local semigroups in Hilbert spaces. Theorem 2.6 gives a sufficient condition for the associated generators of the semigroup to satisfy the condition of Koranyi. In particular our sufficient condition assures the existence of unitary extensions of the semigroup. In section 3 it is shown that this result provides a simple and self-contained proof of the Eskin theorem (which contains that of Devinatz). Other applications and further developments will be discussed elsewhere.

For an analogue theory of local semigroups of symmetric operators related to quantum theory see [21] and [15].

2.-TWO PARAMETERS LOCAL SEMIGROPUS OF ISOMETRIC OPERATORS

Let a, b such that $0 < a, b \leqslant +\infty$.

Let $I = [0,a) \times [0,b) \subset \mathbb{R}^2$.

If s, t $\in$ I we say that $t \leqslant s$ if $s - t \in I$ (it is, for $s = (s_1,s_2)$, $t = (t_1,t_2)$ $t \leqslant s$ if $t_1 \leqslant s_1$,$t_2 \leqslant s_2$)

2.1.DEFINITION Let H be a Hilbert space. A two-parameter local semigroup of isometric operators (shortly T.L.S.I.) is a family of pairs $(S(t), H(t))_{t\in I}$ such that

(i)H(t) is a closed subspace of H , $S(t) : H(t) \to H$ is an isometric operator with domain H(t), such that if s, t $\in$ I and $s \leqslant t$ then $H(t) \subset H(s)$, and $H(0) = H$, $S_0 = id_H$.

(ii)If s, t, s + t $\in$ I then $S(t)\, H(s+t) \subset H(s)$ and $S(s+t)\, f = S(s)\, S(t)\, f$ for all $f \in H(s+t)$.

(iii) $\bigcup_{s>t} H(s)$ is dense in H(t) for all $t \in I$.

(iv)The function $s \mapsto S(s)\, f$ is continuous in $\{ s \in I : s \leqslant t \}$ for all $t \in I$ and each $f \in H(t)$

When we speak about local semigroups of isometric operators (shortly L.S.I.) we understand that they are one-parameter local semigroups of isometric operators $(S(x), H(x))_{x\in[0,a)}$ as defined in [04], (se also [06]), i.e. the family (S(x,y), H(x,y)) where $(x,y) \in [0,a)x[0,+\infty)$ and $H(x,y) = H(x)$, $S(x,y) = S(x)$ satisfies (i)-(iv).

Let $(S(t), H(t))_{t\in I}$ be a T.L.S.I. in the Hilbert space H.

2.2.REMARK. $\forall\ (t_1,r_2) \in I \quad \bigcup_{a\geq s_1>t_1} H(s_1,r_2)$ is dense in $H(t_1,r_2)$, because $\bigcup_{s_1>t_1} H(s_1,r_2) \supset \bigcup_{s>(t_1,r_2)} H(s)$

Also $\forall\ (r_1,t_2) \in I \quad \bigcup_{b\geq s_2>t_2} H(r_1,s_2)$ is dense in $H(r_1,t_2)$.

We will use the following notation:

$S_1(x) = S(x,0) \qquad H^1_x = H(x,0) \qquad x \in [0,a)$

$S_2(y) = S(0,y) \qquad H^2_y = H(0,y) \qquad y \in [0,b)$

It is clear that we have:

2.3.PROPOSITION *The families* $(S_1(x) , H^1_x)_{x\in[0,a)}$ *and* $(S_2(y) , H^2_y)_{y\in[0,b)}$ *are* L.S.I. *in the Hilbert space* H, *called the associated* L.S.I.

Also if $f \in H(x,y)$ *then* $f \in H(x,0) \cap H(0,y)$ *and* $S(x,y) f = S(x,0)\, S(0,y) f = S(0,y)\, S(x,0) f$.

Thus the T.L.S.I. is determined by the two associated L.S.I. and therefore by their infinitesimal generators, provided that H(x,y) is known for all (x,y) (see [06], theorem 3.3).

A_k will denote the infinitesimal generator of (S_k) , k =1, 2. Then (see [04]) $D_k = i\, A_k$ is a symmetric operator in H.

By the two-parameter case of the Stone theorem every strongly continuous group of unitary operators $(U(t))_{t\in\mathbb{R}^2}$, in the Hilbert space H, is of the form

$$U(t) = U(t_1,t_2) = e^{it_1\Omega_1}\, e^{it_2\Omega_2}$$

with Ω_1, Ω_2 selfadjoint commuting operators.

From this and the precedents remarks it follows:

2.4.THEOREM ([22]) *Let* $(S(t), H(t))_{t\in I}$ *be a* T.L.S.I. *in the Hilbert space* H. *Then there exits a strongly continuous group of unitary operators* $(U(t))_{t\in\mathbb{R}^2}$, *in a* *larger Hilbert space such that* $S(t) = U(t)|_{H(t)}$ $\forall\, t \in I$ *if and only if the symmetric operators* D_1 *and* D_2 *have commutative selfadjoint extensions in a larger Hilbert space.*

This theorem reduces the problem of unitary extension of local semigroups to an equally difficult classical problem for the associated pair of generators. This classical problem was studied by several authors (see [03], [16], [22], and for related problems see [19] and [01]). Of these works we shall use the following particularly simple result by Koranyi.

THEOREM ([16]) *Let* $\mathfrak{M}$ *be a dense linear manifold in the Hilbert space* H. *Let* A_o *and* B_o *two symmetric operators defined on* $\mathfrak{M}$; A *and* B *their closures. Let* A *be selfadjoint, and let* B *equal to its restriction to the manifold* $(A + iI)\mathfrak{M}$. *Suppose that the domain of both products* AB *and* BA *contains* $\mathfrak{M}$, *and* AB f = BA f *holds for all* f *in* $\mathfrak{M}$. *Then in an enlarged Hilbert space* $\hat{H} \supseteq H$ *there exists commuting selfadjoint extensions* $\hat{A}$, $\hat{B}$ *of* A *and* B.

The following property is immediate.

2.5.PROPOSITION *Let* $(S(t), H(t))_{t\in I}$ *be a* T.L.S.I. *in the Hilbert space* H. *Then* $\forall\, y \in [0,b)$

$$(S(x,0)|_{H(x,y)} , H(x,y))_{x\in [0,a)}$$

is a L.S.I. *in the Hilbert space* H(0,y).

Also if $x \in [0,a)$ *then*

$$(S(0,y)|_{H(x,y)} , H(x,y))_{y\in [0,b)}$$

is a L.S.I. *in the Hilbert space* H(x,0).

Our main result is the following:

2.6.THEOREM *Let* $(S(t), H(t))_{t\in I}$ *be a* T.L.S.I. *in the Hilbert space* H, $I = [0,a) \times [0,b)$. *Suppose that for every* $y \in [0,b)$ *the* L.S.I. $(S(x,0)|_{H(x,y)} , H(x,y))_{x\in [0,a)}$ *has a unique extension to a strongly continuous group of unitary operators in the Hilbert space* H(0,y).

Then the associated pair of operators D_1 *and* D_2 *satisfaies the hypothesis in the theorem of Koranyi.*

Therefore the T.L.S.I. $(S(t), H(t))_{t\in I}$ *can be extended to a strongly continuous group of unitary operators* $(U(t))_{t\in\mathbb{R}^2}$ *in a larger Hilbert space.*

The proof of this theorem will use the techniques of our previous paper [04].

Before starting the proof of this theorem we need some lemmas. In theses lemmas we will construct a linear manifold $\mathcal{D}^{(1)}$ such that $\mathcal{D}^{(1)}$ is a core of D_1 and D_2, D_1 and D_2 commute in $\mathcal{D}^{(1)}$ and $(D_1 + i\, I)\mathcal{D}^{(1)}$ is a core of D_2. This will allow us to apply the above mentioned result of Koranyi.

2.7.LEMMA *Let* $E = \bigcup_{\substack{0<x<a\\0<y<b}} H(x,y)$ *and let* $\mathcal{D}$ *be the linear manifold spanned by the elements of the form*

$$\frac{1}{r_1\, r_2} \int_0^{r_2} \int_0^{r_1} S_1(x)\, S_2(y)\, f\, dx\, dy ,$$

where $f \in H(r_1^\circ, r_2^\circ) \subset E$, $(r_1^\circ, r_2^\circ) \in (0,a)x(0,b)$ *and* $0 < r_1 < r_1^\circ$, $0 < r_2 < r_2^\circ$

Then:

(a) $\mathcal{D}$ *is a core of* A_1 *and of* A_2.

(b) $\mathcal{D} \subset \text{dom}\,(A_1 A_2)$, $\mathcal{D} \subset \text{dom}\,(A_2 A_1)$ *and*

$A_1 A_2 h = A_2 A_1 h \quad \forall\, h \in \mathcal{D}$

Moreover, if

$$h = \frac{1}{r_1\, r_2} \int_0^{r_2} \int_0^{r_1} S_1(x)\, S_2(y)\, f\, dx\, dy \;\in \mathcal{D}$$

then

$$A_1 A_2 h = A_2 A_1 h = \frac{1}{r_1\, r_2} (S_1(r_1) - I)(S_2(r_2) - I) f$$

$$= \frac{1}{r_1\, r_2} (S_2(r_2) S_1(r_1) f - S_1(r_1) f - S_2(r_2) f + f)$$

Proof:

(a)Let's consider the A_1 case.

We have (see [04])

$$\mathrm{dom}(A_1) = \left\{ f \in \bigcup_{x\in(0,a)} H(x,0) \mid \lim_{t\to 0^+} (S_1(t) f - f)/t \text{ exists} \right\}$$

Let $f \in \mathrm{dom}(A_1)$. Then there exists $s_0 \geq 0$ such that $f \in H(s_0,0)$.

Let $\{ r_1(n) \}$ be a sequence of real numbers such that $0 < r_1(n) < s_0/2$ and $\lim_{n\to+\infty} r_1(n) = 0$.

Then (see [04])

$$\lim_{n\to+\infty} \frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x) f \, dx = f , \quad \text{and}$$

$$\lim_{n\to+\infty} A_1 \left(\frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x) f \, dx \right) =$$

$$= \lim_{n\to+\infty} (S_1(r_1(n)) f - f)/r_1(n) = A_1 f .$$

Let $f_n = \dfrac{1}{r_1(n)} \displaystyle\int_0^{r_1(n)} S_1(x) f \, dx$.

Since $r_1(n) < s_0/2$ we have that

$$S_1(x) f \in H(s_0 - r_1(n),0) \subset H(s_0/2,0) \text{ if } 0 \leqslant x \leqslant r_1(n)$$

and $\bigcup_{0<t<b} H(s_0/2,t)$ is dense in $H(s_0/2,0)$.

Therefore there exists a sequence $\{t_n\}$, $0 < t_n < b$, and a sequence $\{ g_n \} \subset H(s_0/2,t_n)$ such that

$$\| g_n - f \| < (r_1(n))^2$$

Let $\{ r_2(n) \}$ be such that

$0 < r_2(n) < t_n$, $\lim_{n\to+\infty} r_2(n) = 0$ and

$$\left\| \frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y) g_n \, dy - g_n \right\| < (r_1(n))^2$$

Let $h_n = \frac{1}{r_1(n)\ r_2(n)} \int_0^{r_2(n)} \int_0^{r_1(n)} S_1(x)\ S_2(y)\ g_n\ dx\ dy$

then $h_n \in \mathcal{D}$ and

$$\| h_n - f \| \leq$$

$$\leq \| \frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x) \left[\frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y)\ g_n\ dy - g_n \right] dx \| +$$

$$+ \quad \| \frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x)\ (\ g_n - f\ )\ dx \|$$

$$+ \quad \| \frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x)\ f\ dx \ - \ f \|$$

$$\leq \| \frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y)\ g_n\ dy - g_n \| + \| g_n - f \|$$

$$+ \quad \| \frac{1}{r_1(n)} \int_0^{r_1(n)} S_1(x)\ f\ dx \ - \ f \|$$

Clearly these three last terms tend to 0 as n tends to $+\infty$.

$$A_1 h_n - A_1 f_n =$$

$$= \frac{1}{r_1(n)}\ (\ S_1(r_1(n)) - I\ )\ (\ \frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y)\ g_n\ dy - f\ )$$

$$= \frac{1}{r_1(n)}\ (\ S_1(r_1(n)) - I\ )\ (\ \frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y)\ g_n\ dy - g_n\ )$$

$$+ \quad \frac{1}{r_1(n)}\ (\ S_1(r_1(n)) - I\ )\ (\ g_n - f\ )$$

Threrefore

$$\| A_1 h_n - A_1 f_n \| \leq (\frac{2}{r_1(n)})\ (\ r_1(n)\ )^2 + (\frac{2}{r_1(n)} \quad (\ r_1(n)\ )^2$$

Since $A_1 f_n \to A_1 f$ if $n \to +\infty$ we have

$$h_n \to f \quad , \quad A_1 h_n \to A_1 f \quad \text{if} \quad n \to +\infty$$

So $\mathcal{D}$ is a core of A_1 .

(b) follows from the general theory of L.S.I. (see [04])

2.8.LEMMA *Let* $\mathcal{D}^{(1)}$ *be the linear manifold spanned by the elements of the form*

$$\frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} S_1(x)\, S_2(y)\, f\, dx\, dy$$

where $f \in H(r_1^{\circ}, r_2^{\circ}) \subset \mathcal{D}$, $(r_1^{\circ}, r_2^{\circ}) \in (0,a)x(0,b)$ *and* $0 < r_1 < r_1^{\circ}$, $0 < r_2 < r_2^{\circ}$.

Then:

(a) $\mathcal{D}^{(1)}$ *is a core of* A_1 *and of* A_2 .

(b) $\mathcal{D}^{(1)} \subset \text{dom}\,(A_1 A_2)$, $\mathcal{D}^{(1)} \subset \text{dom}\,(A_2 A_1)$, $A_1 A_2 h = A_2 A_1 h \quad \forall\, h \in \mathcal{D}^{(1)}$, *and the last assertion of the preceding lemma is true.*

Proof:

(b) Is clear.

(a) It is sufficient to see that $\mathcal{D}^{(1)}$ is core of $A_1|_{\mathcal{D}}$.

Let $f \in \mathcal{D}$. It is clear that $f \in E$, so $f \in H(r_1^{\circ}, r_2^{\circ})$ for some $r_1^{\circ}, r_2^{\circ} > 0$. Let $\{ r_1(n) \}$ and $\{ r_2(n) \}$ be two sequences of positive real number converging to 0 such that $r_1(n) < r_1^{\circ}$, $r_2(n) < r_2^{\circ}$ and let

$$f_n = \frac{1}{r_1(n)\, r_2(n)} \int_0^{r_2(n)} \int_0^{r_1(n)} S_1(x)\, S_2(y)\, f\, dx\, dy$$

Then $f_n \to f$ and

$$A_1 f_n = \frac{1}{r_2(n)} \int_0^{r_2(n)} S_2(y) \left[\frac{1}{r_1(n)} (S_1(r_1(n)) f - f) \right] dy \to A_1 f$$

if $n \to +\infty$

Therefore $\mathcal{D}^{(1)}$ is a core of A_1.

As already observed (see proposition 2.5), if $r \in [0,b)$ then $(S(x,0)|_{H(x,r)} , H(x,r))_{x \in [0,a)}$ is a L.S.I. in the Hilbert space $H(0,r)$.

Let's denote its infinitesimal generator by $A_1(r)$. It is clear that

$$A_1(r) = A_1 |_{dom(A_1(r))}$$

whrere

$$\begin{aligned} dom\,(A_1(r)) &= \{ f \in dom(A_1) : f \in H(0,r) \} \\ &= \left\{ f \in \bigcup_{x \in (o,a)} H(x,r) \mid \text{existe} \lim_{t \to o^+} (S_1(t) f - f)/t \right\} \end{aligned}$$

In a similar way the following two lemmas can be proved.

2.9.LEMMA *Let $\mathcal{D}(r)$ be the linear manifold spanned by the elements of the form:*

$$\frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} S_1(x) S_2(y) f \, dx \, dy$$

where $f \in H(r_1^\circ , r + r_2^\circ) \subset \mathcal{D}$, *and* $0 < r_1 < r_1^\circ$, $0 < r_2 < r_2^\circ$.

Then $\mathcal{D}(r)$ is a core of $A_1(r)$.

2.10.LEMMA *Let $\mathcal{D}^{(1)}(r)$ be the linear manifold spanned by the elements of the form:*

$$\frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} S_1(x) S_2(y) f \, dx \, dy$$

where $f \in H(r_1^\circ , r + r_2^\circ) \subset \mathcal{D}$, *and* $0 < r_1 < r_1^\circ$, $0 < r_2 < r_2^\circ$.

Then $\mathcal{D}^{(1)}(r)$ is a core of $A_1(r)$.

2.11.REMARK By induction and the precedent results the following generalization of lemma 2.8 can be proved:

Let $\mathcal{D}^{(0)} = \mathcal{D}$, and for $n \geq 1$ let $\mathcal{D}^{(n)}$ be the linear manifold spanned by the elements of form

$$\frac{1}{r_1\ r_2} \int_0^{r_2} \int_0^{r_1} S_1(x)\ S_2(y)\ f\ dx\ dy$$

where $f \in H(r_1^\circ , r_2^\circ) \bigcap \mathcal{D}^{(n-1)}$, and $0 < r_1 < r_1^\circ$, $0 < r_2 < r_2^\circ$.

Then:

(a) $\mathcal{D}^{(n)}$ is a core of A_1 and of A_2 .

(b) $\mathcal{D}^{(n)} \subset \text{dom}\ (A_1\ A_2)$, $\mathcal{D}^{(n)} \subset \text{dom}\ (A_2\ A_1)$, $A_1\ A_2\ h = A_2\ A_1\ h \quad \forall\ h \in \mathcal{D}^{(n)}$, and the last assertion of lemma 2.7. is also true.

The corresponding generalization of lemma 2.10 is also true.

2.12.COROLLARY *If for some* $r \in [0,b)$ *the* L.S.I. $(S(x,0)|_{H(x,r)}\ ,\ H(x,r)\ )_{x \in [0,a)}$ *has a unique extension, to a strongly continuous group of unitary operators in the Hilbert space* $H(0,r)$ *then the linear manifold* $(A_1 + I)\ (\mathcal{D}^{(1)}(r))$ *is dense in* $H(0,r)$ $\forall\ n \geq 1$.

Proof:

In this case the operator $i\ A_1(r)$ must have selfadjoint closure (see [04]). Since $\mathcal{D}^{(1)}(r)$ is a core for this operator the desired the result follows.

Proof of the theorem 2.6.:

We only need to prove that $(D_1 + i\ I)\mathcal{D}^{(1)}$ is a core of D_2 because then from the above mentioned theorem of Koranyi it will follow that D_1 and D_2 have selfadjoint commutative extensions to a larger Hilbert space, yielding the desired result.

Let $\mathfrak{G} = (D_1 + i\ I)\mathcal{D}^{(1)}$ and let $C_2 = D_2\ |_{\mathfrak{G}}$.

For $f \in \mathcal{D} = \mathcal{D}^{(o)}$ and for r_1 and r_2 positive and sufficiently small let

$$M(r_1,r_2)\, f = \frac{1}{r_1 r_2} \int_0^{r_2} \int_0^{r_1} S_1(x)\, S_2(y)\, f \, dx\, dy$$

Then we have

$$< C_2^* g, (D_1 + i\, I)\, M(r_1,r_2) f > \; = \; < g, D_2 (D_1 + i\, I)\, M(r_1,r_2) f >$$

$\forall\, g \in \text{dom}\,(C_2^*)$, $\forall\, f \in \mathcal{D}$, r_1 and r_2 small.

The first member of this equality is equal to

$$< C_2^* g\ ,\ \frac{i}{r_1 r_2} \left[\int_0^{r_2} S_2(y)\, (\, S_1(r_1)\, f - f\,)\, dy + \int_0^{r_2} \int_0^{r_1} S_1(x)\, S_2(y)\, f\, dx\, dy \right] > \; =$$

$$= \; < \int_0^{r_2} S_2^*\,(y)\, C_2^* g\, dy\ ,\ \frac{i}{r_1 r_2} \left[S_1(r_1)\, f - f + \int_0^{r_1} S_1(x)\, f\, dx \right] >$$

The second member is equal to

$$< g, -\frac{1}{r_1 r_2}\, (S_2(r_2) - I) \left[(S_1(r_1) - I) f - \int_0^{r_1} S_1(x)\, f\, dx \right] > \; =$$

$$= - < \frac{1}{r_2}\, (S_2^*\,(r_2)\, g - g)\ ,\ \frac{1}{r_1} \left[(S_1(r_1) f - f) + \int_0^{r_1} S_1(x)\, f\, dx \right] >$$

Letting $r_2 \to o^+$ we obtain

$$< \frac{1}{r_2} \int_0^{r_2} S_2^*\,(y)\, C_2^* g\, dy\ ,\ i\, (A_1 f + f) > \; =$$

$$= - < \frac{1}{r_2}\, (S_2^*\,(r_2)\, g - g)\ ,\ (A_1 f + f) >$$

if $f \in \mathcal{D}^{(1)}(r_2)$.

or, what is the same

$$< \frac{1}{r_2} \int_0^{r_2} S_2^*(y)\, C_2^* g\, dy , \quad (A_1 f + f) > =$$

$$= - < \frac{i}{r_2} (S_2^*(r_2)\, g - g) , (A_1 f + f) >$$

if $f \in \mathcal{D}^{(1)}(r_2)$.

Corollary 2.12 implies that

$$< \frac{1}{r_2} \int_0^{r_2} S_2^*(y)\, C_2^* g\, dy , h > = - < \frac{i}{r_2} (S_2^*(r_2)\, g - g) , h >$$

for all $h \in H(0,r_2)$.

Then

$$\lim_{r_2 \to o^+} < \frac{i}{r_2} (S_2^*(r_2)\, g - g) , h > =$$

$$= - \lim_{r_2 \to o^+} < \frac{1}{r_2} \int_0^{r_2} S_2^*(y)\, C_2^* g\, dy , h >$$

$$= - < C_2^* g, h > \ \forall\, h \in \bigcup_{r>o} H(0,r)$$

Let $f \in \mathrm{dom}\,(D_2)$ (remember that $\mathrm{dom}(D_2) \subset \bigcup H(0,r)$)

Then

$$< C_2^* g, f > = \lim_{r_2 \to o^+} - < \frac{i}{r_2} (S_2^*(r_2)\, g - g) , f >$$

$$= \lim_{r_2 \to o^+} - < i\, g , \frac{1}{r_2} (S_2(r_2)\, f - f) >$$

$$= - < i\, g , A_2 f >$$

$$= < g , D_2 f >$$

Therefore

$$< C_2^* g, f > = < g , D_2 f > \quad \forall\, g \in \mathrm{dom}(C_2^*) , \ \forall\, f \in \mathrm{dom}(D_2)$$

So $C_2^* \subset D_2^* \Rightarrow \overline{D_2} \subset C_2^{**} = \overline{C_2} \Rightarrow \mathfrak{S}$ is a core of D_2.

2.13.REMARK Theorem 2.6 is also true if we only suppose that there exists a sequence $\{ y_n \} \subset (0,b)$

converging to 0, such that for every n the L.S.I. $(S(x,0)|_{H(x,y_n)} , H(x,y_n))_{x\in[0,a)}$ has a unique extension to a strongly continuous group of unitary operators in the Hilbert space $H(0,y_n)$.

3.-THE KREIN-ESKIN THEOREM

The function $k : (-2a,2a) \to \mathbb{C}$ ($0 < a < +\infty$) is said to be positive definite if for every subset $\{ x_p \}_{p=1}^{n} \subset (-a,a)$ and every set $\{ c_p \}_{p=1}^{n}$ of complex numbers

$$\sum_{p=1}^{n} \sum_{q=1}^{n} c_p \overline{c_q} k(x_p - x_q) \geq 0 \tag{3.0}$$

holds.

M. G. Krein [17] proved that if $k : (-2a,2a) \to \mathbb{C}$ is a continuous and positive definite function then k has a continuous and positive definite extension to the whole real line.

It is natural to consider the following problem:

Let $Q = (-a,a)\times(-b,b)$ ($0 < a, b \leq +\infty$) be an open rectangle in the (x,y)-plane and let $k : 2Q \to \mathbb{C}$ be a continuous and positive definite function (i.e. (3.0) is satisfied in Q). When does there exists a continuous and positive definite extension of k to the whole plane ?

In 1944 M. Livshitz [18] proved that the answer is affirmative if a or b are equal to $+\infty$.

The more general situation where $Q = G \times (-a,a)$, G a locally compact abelian group was studied in [11] and [10].

In 1959 A. Devinatz obtained the following result:

THEOREM ([07]). *Let* k *and* Q *be as before. If* k(x,o) *and* k(o,y) *each have unique continuous and positive definite extensions along the* x*-axis and the* y*-axis respectively, then* k *has a unique continuous and positive definite extension to the whole plane.*

In 1960 G.I. Eskin ([09]) proved the following refinement of the existence assertion of the Devinatz theorem:

THEOREM ([09]) *Let* k *and be* Q *as before. If one of the functions* k(x,o) *or* k(o,y) *has a unique continuous positive* *definite extension to the real axis, then* k *has a, not necessarily unique, continuous and positive definite extension to the whole plane.*

A different approach to this result is given in Berezanski's book [03].

In 1963 W. Rudin ([23]) showed that the answer to the problem is, in general, negative.

See also [13], [14].

In this part of the paper we will apply theorem 2.6. to prove the bidimensional version of the Krein theorem due to Eskin [09]. Let us recall that we are going to prove the following result:

3.1.THEOREM ([09]). *Let* $Q = (-a, a) \times (-b, b)$ $(0 < a,b < +\infty)$ *be an open rectangle in the* (x,y)*-plane and let* $k : 2Q \to \mathbb{C}$ *be a continuous and positive definite function.*

Suppose that one of the restrictions $k(x,o)$ *or* $k(o,y)$ *has only one continuous and positive definite extension to the whole real axis*

Then the function k has a continuous and positive definite extension to the whole plane.

Following the idea in [04] we first introduce a T.L.S.I. associated to k and study some of its properties which are of independent interest.

Let $K : 2Q \times 2Q \to \mathbb{C}$ be the function defined by

$$K (s,t) = k (s - t)$$

Then K is a positive definite Toeplitz kernel in Q, it is , K is continuous, K (s,t) depends only on the difference s - t, and

$$(3.2) \qquad \sum_{s,t\in Q} K (s,t) C(s) \overline{C(t)} \geq 0$$

whenever $C : Q \to \mathbb{C}$ is a finite support function.

For $t \in Q$ let $K_t : Q \to \mathbb{C}$ be the function defined by $K_t(s) = K(s,t)$ and let E be the linear space defined by

$$E = \left\{ f:Q \to \mathbb{C} \mid f = \sum_{k=1}^{n} a_k K_{t(k)} ,\ n \in \mathbb{N},\ a_k \in \mathbb{C},\ t(k) \in Q \right\}$$

Clearly E elements are continuous functions.

If $a = \sum_{k=1}^{n} a_k K_{t(k)}$, $b = \sum_{j=1}^{m} b_j K_{s(j)}$ are elements of E, we define

(3.3) $\langle a , b \rangle = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k \overline{b}_j K (s(j),t(k))$

Then $\langle \, , \, \rangle$ is a positive semidefinite sesquilinear form in E, and

(3.4) $a(s) = \langle a , K_s \rangle \; \forall a \in E \; \forall s \in Q$

Therefore we have

(3.5) $|a(s)| \leq \|a\| \, \| K_s \| = \|a\| \, (K(o,o))^{1/2}$
$\forall a \in E \;\; \forall s \in I$

Let $H = H_k$ be the completion of E. Then H elements are continuous functions, and if we continue using $\langle \, , \, \rangle$ for the product in H, 3.4 and 3.5 remain true for H elements. So K is a reproducing kernel for H (see [02]).

Let $I = [0,a) \times [0,b)$ and for $t \in I$ let

$$E_t = \left\{ h \in E \mid h = \sum_{k=1}^{n} a_k K_{t(k)} , \; n \in \mathbb{N}, \; a_k \in \mathbb{C}, \; t(k), \; t(k) + t \in Q \right\}$$

If $h = \sum_{k=1}^{n} a_k K_{t(k)} \in E_t$ we define

$$S_t h = \sum_{k=1}^{n} a_k K_{t(k)+t}$$

If $h, g \in E_t$ then we have

$$(S_t h , S_t g) = (h , g)$$

Moreover, if H_t is the closure of E_t in H then it is clear that S_t can be extended to a lineal isometry from H_t in H and it is easy to verify that $(S_t,H_t)_{t\in I}$ is a T.L.S.I. in H.

We will use the notations of section 2.

Let $r \in [0,b)$. As observed in section 2 $(S(x,0)|_{H(x,r)} , H(x,r))_{x\in[0,a)}$ is a T.L.S.I. in the Hilbert space $H(0,r)$.

3.6. DEFINITION *Let* $W_1(r)$ *be the operator in* $H(0,r)$ *defined by*

$\operatorname{dom}(W_1(r)) = \{ h \in H(0,r) \mid \frac{\partial h}{\partial x}(x,y)$ *exists for* $x \in (-a,a)$ *and* $\frac{\partial h}{\partial x}(x,y) = g(x,y)$ *for some* $g \in H(0,r) \}$

and $W_1(r)\, h = -i \frac{\partial h}{\partial x}$ *if* $h \in \operatorname{dom}(W_1(r))$.

We have:

3.7.THEOREM $(i A_1(r))^* = W_1(r)$

The the proof of this theorem is very similar to the analogous theorem for the one dimensional case and we only sketch the main steps.

Given $x \in (-a,a)$ and $y \in (-b,b)$ let

$$f_{x,y} = \frac{1}{\lambda} \int_0^{\lambda} K_{(x-\lambda,y)}\, d\lambda$$

for λ sucht that $x-\lambda \in (-a,a)$. In the same way as in the one dimensional case (see lemma 2 of [04]) one proves that

$f_{x,y} \in \operatorname{dom}(A_1(r))$ and $A_1(r)\, f_{x,y} = (K_{(\lambda,y)} - K_{(\lambda-x,y)})/x$

and also that if $h \in \operatorname{dom}(A_1(r))$ then

$\frac{\partial h}{\partial x}(x,y)$ exists for $x \in (-a,a)$ and $A_1(r)^*\, h = \frac{\partial h}{\partial x}$

Therefore $(i A_1(r))^* \subset W_1(r)$

Since convergence in H implies uniform convergence we have that $W_1(r)$ is a closed operator.

Using the same idea of lemma 3 of [04] one proves that $i A_1(r) \subset W_1(r)^*$. So $(i A_1(r))^* = W_1(r)$

3.8.COROLLARY $W_1(r)^*$ *is a symmetric operator.*

The deficiency indices of $iA_1(r)$ are

$m = \dim \operatorname{kernel} ((i A_1(r))^* + z I)$ if $\operatorname{Im} z > o$

$n = \dim \operatorname{kernel} ((i A_1(r))^* + z I)$ if $\operatorname{Im} z < o$

It is easy to see that the function $J:H(0,r) \to H(0,r)$ defined by $J\, h\, (x,y) = h(-x,y)$ is a conjugation (i.e. $J(h+g) = Jh + Jg$, $J(ah) = \bar{a}Jh$, $J^2 = I$, and $\langle Jh, Jg\rangle = \langle g, h\rangle$. Also it is clear that $W_1(r)\, J = J\, W_1(r)$. From the general theory of symmetric operators, see [08], it follows that the deficiency indices of $iA_1(r)$ are equal.

The general solution of the equation $i\frac{\partial h}{\partial x} + zh = 0$ is $h(x,y) = h(0,y)\,e^{izx}$.

So by theorem 3.3. of [06] we have

3.9.THEOREM *The following conditions are equivalent:*

(a)The L.S.I. $(S(x,0)|_{H(x,r)}, H(x,r))_{x\in[0,a)}$ *has a unique extension to a strongly continuous group of unitary operators in* $H(0,r)$.

(b)Any non zero function of the form $h(x,y) = h(0,y)\,e^{izx}$ $(\mathrm{Im}(z) \neq 0)$ *is in* $H(0,r)$.

(c)Any non zero function of the form $h(x,y) = h(0,y)\,e^{-x}$ *is in* $H(0,r)$.

Let H_1 be the reproducing Hilbert space corresponding to $K(x_1-x_2, 0)$. Since Q is a rectangle (see [02]), H_1 is the set of the restrictions of the elements of H to any line parallel to the x-axis. Therefore if there is some non zero function of the form $h(x,y) = h(0,y)\,e^{-x}$ in H then there will exists $y_o \in (-b,b)$ such that $h(0,y_o) \neq 0$ and the function $f(x) = e^{-x}$ will be in H_1. So, if the kernel $K(x_1-x_2, 0)$ has a unique positive definite extension to the real line no non zero function of the form $h(x,y) = h(0,y)\,e^{-x}$ can be in H (see [04]), and therefore can not be in any $H(0,r)$. We have proved:

3.10.**LEMMA** *Let* $Q = (-a, a) \times (-b, b)$ $(0 < a,b < +\infty)$ *be an open rectangle in the* (x,y)*-plane and let* $k : 2Q \to \mathbb{C}$ *be a continuous and positive definite function. Let* $(S_t,H_t)_{t\in I}$ *be the* T.L.S.I. *corresponding with* k. *If the restriction of* k *to the* x*-axis has a unique continuous positive definite extension to the whole real line then, each of the* L.S.I. $(S(x,0)|_{H(x,r)}, H(x,r))_{x\in[0,a)}$ *in* $H(0,r)$ $(r \in [0,b))$, *has a unique extension to a strongly continuous group of unitary* *operators in* H(0,r).

Proof of theorem 3.1:

It is clear that we can suppose that k(x,o) has a unique continuous positive definite extension to the real line.

Let $(S_t, H_t)_{t\in I}$ be the T.L.S.I. in the Hilbert space H, associated with k. By lemma 3.10 each of the L.S.I. $(S(x,0)|_{H(x,r)} , H(x,r))_{x\in[0,a)}$ in H(0,r) ($r \in [0,b)$) has a unique extension to a strongly continuous group of unitary operators in H(0,r).

From theorem 2.6. it follows that the T.L.S.I. $(S(t), H(t))_{t\in I}$ can be extended to a strongly continuous group of unitary operators $(U(t))_{t\in\mathbb{R}^2}$ in a larger Hilbert space F.

Let $\tilde{k}(x,y) = \langle K_{(o,o)} , U_{(x,y)} K_{(o,o)} \rangle_F \quad (x,y) \in \mathbb{R}^2$.

It is clear that $\tilde{k}$ is continuous.

Let $(x,y) \in Q$.

If $(x,y) \in I$ then

$$\begin{aligned} k(x,y) &= \langle K_{(o,o)} , K_{(x,y)} \rangle = \\ &= \langle K_{(o,o)} , S_{(x,y)} K_{(o,o)} \rangle \\ &= \langle K_{(o,o)} , U_{(x,y)} K_{(o,o)} \rangle \\ &= \tilde{k}(x,y) . \end{aligned}$$

If $x \geqslant 0$ and $y < 0$.

$$\begin{aligned} k(x,y) &= \langle K_{(o,-y)} , K_{(x,o)} \rangle = \\ &= \langle S_{(o,-y)} K_{(o,o)} , S_{(x,o)} K_{(o,o)} \rangle \\ &= \langle U_{(o,-y)} K_{(o,o)} , U_{(x,o)} K_{(o,o)} \rangle \\ &= \langle K_{(o,o)} , U_{(x,y)} K_{(o,o)} \rangle \\ &= \tilde{k}(x,y) . \end{aligned}$$

Continuing in this way we can prove that $\tilde{k}$ is an extensión of k, and it is straightforward to see that $\tilde{k}$ is positive definite.

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Dpto. de Matematica.
Facultad de Ciencias.
Universidad Central de Venezuela

Mailing address:
Apartado 47686
Los Chaguaramos
Caracas 1041-A
VENEZUELA

MSC numbers: Primary 47D03, Secondary 42A82

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