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INTEGRAL REPRESENTATION OF KERNELS INVARIANT WITH RESPECT TO LOCAL SEMIGROUPS OF ISOMETRIES AND GENERALIZATION OF THE KREIN-SCHWARTZ THEOREM

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ABSTRACT

We show that the theory of local semigroups of isometric operators leads to a generalization of the Krein-Berezanski-Korsunski decomposition of invariant kernels, which allows to extend the Bochner-Schwartz and the Krein-Schwartz theorems to generalized Toeplitz kernels. As a corollary the Nehari theorem in R is derived.

1. INTRODUCTION

1.1. Gelfand's triplets and the Berezanski lemma

Let H_0 be a (separable) Hilbert space with scalar product $(,)_0$, H_+ a dense subspace of H_0 and $(,)_+$ a scalar product in H_+ such that H_+ is a Hilbert space under $(,)_+$ and $||u||_0 \leq ||u||_+$ $\forall$ $u \in H_+$. Let H_- be the antidual of H_+, and for each $f \in H_0$ let α_f be the antilinear functional in H_+ defined by $\alpha_f(u) = (f,u)_0$, $\forall$ $u \in H_+$, so that $||\alpha_f|| \leq ||f||_0$ and $\alpha_f = \alpha_g$ implies $f=g$. Then $f \to \alpha_f$ is a continuous injection of H_0 into H_- and we can write

$$(1.1) \qquad H_+ \subset H_0 \subset H_-$$

where H_+ is dense in H_0, H_0 is dense in H_-, the inclusions 0: $H_+ \to H_0$ and 0^+: $H_0 \to H_-$ have norm less or equal than one and $0^+ f = \alpha_f$. Such a system (1.1) is called Gelfand triplet.

It is easy to see that there is an isometric operator I from H_- onto H_+ such that $0^* = I|_{H_0}$ and $0^+ = I^{-1} 0^*$, so that 0 is a Hilbert-Schmidt operator if and only if 0^+ is such an operator, and in this case the triplet (1.1) is said to be quasinuclear.

Let $H_+ \subset H_0 \subset H_-$ be a Gelfand triplet. For $\alpha \in H_-$ and $u \in H_+$ we define $(\alpha,u)_0 = \alpha(u)$, and say that an operator $A: H_+ \to H_-$ is non-negative if $(Au,u)_0 = (Au)(u) \geq 0$ $\forall$ $u \in H_+$, define

$$\text{trace } (A) = \sum (Ae_j, e_j),$$ where $\{e_j\}$ is an orthonormal basis in H_+.

The following result is a slight variant of a lemma of Berezanski [6] (cfr Korsunski [12]).

Berezanski's lemma

Let $H_+ \subset H_0 \subset H_-$ be a Gelfand triplet, and θ an operator-valued measure in R assigning to each Borel set $\Delta \subset R$ a non-negative operator $\theta(\Delta)$: $H_+ \to H_-$ such that

$$\rho(\Delta) = \text{trace } \theta(\Delta) < +\infty.$$

Then ρ is a finite Borel measure on R and for ρ-almost every λ in R there exists a non-negative operator $P(\lambda)$: $H_+ \to H_-$ such that $||P(\lambda)|| \leq \text{trace } (P(\lambda)) = 1$ and

$$\theta(\Delta) = \int_\Delta P(\lambda)\, d\rho(\lambda) \quad \text{for every Borel set } \Delta \subset R.$$

Moreover $P(\lambda)$ is uniquely determined for ρ-almost every point λ.

1.2. Local Semigroups of operators

We will use the following results and definitions from [7].

Definition:

We say that $(S_t,H_t)_{t\in(o,a)}$, $o<a\leq+\infty$ is a local semigroup of operators (L.S.O.) on the Hilbert space H if:

(i) For each r, $o<r<a$, H_r is a closed linear subspace of H, such that $H_r \subset H_s$ if $o<s<r<a$, and S_r: $H_r \to H$ is a linear bounded operator.

(ii) If r, $t \in (o,a)$, $r+t<a$ then $S_t(H_{r+t}) \subset H_r$ and $S_{r+t}f = S_r S_t f$ $\forall f \in H_{r+t}$.

(iii) $\lim_{t\to o} || S_t f - f || = 0$ $\forall f \in \bigcup_{r\in(0,a)} H_r$

(iv) $\bigcup_{r\in(o,a)} H_r$ is dense in H.

(v) $\bigcup_{r\in(r_o,a)} H_r$ is dense in H_{r_0} $\forall r_0 \in (o,a)$.

If all the S_r: $H_r \to H$ are isometric operators then $(S_t,H_t)_{t\in(o,a)}$ is said to be a local semigroup of isometric operators. (L.S.I.).

The infinitesimal generator of a L.S.O is defined as

$$Af = \lim_{t\to o^+} \frac{S_t f - f}{t}$$

for $f \in D(A) = \{f \in \bigcup_{r\in(o,a)} H_r \mid \lim_{t\to o^+} \frac{S_t f - f}{t} \text{ exists} \}$.

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The basic result is the following extension theorem: If $(S_t,H_t)_{t\in(o,a)}$ is a L.S.I. on the Hilbert space H, then D(A) is dense in H, iA is a symmetric operator, and there exists a Hilbert space G containing H as a closed subspace and a strongly continuous group of unitary operators $(U_t)_{-\infty<t<+\infty}$ on G such that $S_t = U_t|_{H_t}$ $\forall t\in(o,a)$.

Moreover a strongly continuous group of unitary operators extends the L.S.I. if and only if the infinitesimal generator of the group is an extension of the generator of the semigroup.

We will say that F is a generalized spectral measure of the L.S.I. (S_t,H_t) if F is a generalized spectral measure of iA (see [1]).

The extension theorem leads then to the following spectral representation of the L.S.I.:

$$F(\Delta)S_t = \int_\Delta e^{it\lambda}\, dF(\lambda) \text{ if } t\in(o,a) \text{ and } \Delta\subset R \text{ is a Borel set.}$$

2. LOCAL FORM OF THE THEOREM OF GELFAND-KOSTUCHENKO-BEREZANSKI-KORSUNSKI

Let $H_+ \subset H_0 \subset H_-$ be a Gelfand triplet, let $(S_t,H_t)_{t\in(o,a)}$ be a L.S.I. on the Hilbert space H_0 and suppose that there exists a dense linear subspace D of H_+ such that if $D_t = D\cap H_t$ then D_t is dense in H_t and $S_t(D_t)\subset D$ $\forall t\in(o,a)$. Then we will say that the L.S.I. is equipped by the triplet and D.

We say that $\alpha\in H_-$ is a generalized eigenvector of the L.S.I. with eigenvalues e_λ ($e_\lambda(t) = e^{i\lambda t}$) if $(\alpha,S_tu)_0 = e^{i\lambda t}(\alpha,u)_0$ $\forall u\in D_t$ $\forall t\in(o,a)$.

An operator P: $H_+\to H_-$ is called a generalized projection on an eigenspace with eigenvalues e_λ, if P is a Hilbert-Schmidt operator, $P\geq 0$, $\|P\|\leq \text{trace}(P) = 1$ and $\forall u\in H_+$ we have that $Pu\in H_-$ is a generalized eigenvector of the L.S.I. with eigenvalues e_λ.

Similarly is defined the notion of generalized eigenvectors of self-adjoint operator, and a classical theorem of Gelfand-Kostuchenko asserts that every self-adjoint operator equipped by quasinuclear triplet posseses "many" generalized eigenvectors. Berezanski [6] extended this theorem to symmetric operators (with simplified proof), and Korsunski [12] gave a version of this theorem for unitary groups. The following theorem extends that of Korsunski to local semigroups of isometries.

<u>Theorem 1</u>

Let $H_+ \subset H_0 \subset H_-$ be a quasinuclear Gelfand triplet, $(S_t,H_t)_{t\in(o,a)}$ a L.S.I. on H_0 equipped by the triplet and the subspace $D\subset H_+$, and F a generalized spectral measure of the L.S.I..

Then

$$0^+F(\Delta)\,0 = \int_\Delta P(\lambda)\, d\rho(\lambda)$$

for every Borel set $\Delta\subset R$, where $\rho\geq o$ is a finite Borel measure in R and for ρ-almost every λ, $P(\lambda)$: $H_+\to H_-$ is a generalized projector operator on the eigenspace with eigenvalues e_λ. For $\Delta= R$ the above theorem gives an expansion of the identity operator in terms of generalized projections on generalized eigenspaces.

<u>Proof</u>:

Since 0 and 0^+ are Hilbert-Schmidt operators we have that for every Borel set $\Delta\subset R$ $\theta(\Delta) = 0^+F(\Delta)0$ is nuclear, that is $\rho(\Delta) = \text{trace}\ \theta(\Delta) < +\infty$.

By Berezanski's lemma we have

$$0^+F(\Delta)\,0 = \int_\Delta P(\lambda)\, d\rho(\lambda)$$

with $P(\lambda)\geq 0$, $\|P(\lambda)\|\leq \text{trace}\ P(\lambda) = 1$ for ρ-almost every λ.

Let

$$v\in H_+,\ t\in(o,a) \text{ and } u\in D_t.$$

Then, using the spectral representation of the L.S.I. we get that

$$\int_\Delta (P(\lambda)v,\ S_tu)_0\, d\rho(\lambda) = (0^+F(\Delta)\,0v,\ 0S_tu)_0$$
$$= (F(\Delta)\,0v,\ 0S_tu)_0 =$$
$$= (0v,\ F(\Delta)\,S_t0u)_0$$
$$= \int e^{it\lambda}(0v,F(d\lambda)0u)_0 =$$
$$= \int e^{it\lambda}(0^+F(d\lambda)0v,u)_0 =$$
$$= \int e^{it\lambda}(P(\lambda)v,u)_0\, d\rho(\lambda).$$

Since the last equality holds for every Borel set $\Delta\subset R$ we have that there exist a Borel set $R_{uv}\subset R$ with $\rho(R-R_{uv}) = 0$ such that $(P(\lambda)v,\ S_tu)_0 = e^{it\lambda}\,(P(\lambda)v,u)_0$ $\forall\lambda\in R_{uv}$. From this and the fact that the Hilbert space is separable the result follows.

3. LOCAL FORM OF THE THEOREM OF KREIN-BEREZANSKI-KORSUNSKI

Through this section H_+ will be a Hilbert space with an involution $u \to \bar{u}$.

A kernel in H_+ is a bounded bilinear form B: $H_+\times H_+ \to \mathbb{C}$.

We will say that B is positive definite (pd) if $B(u,\bar{u})\geq 0$ $\forall u\in H_+$. In this case there is an associated non-negative operator $\tilde{B}$: $H_+\to H_+$ such that $(\tilde{B}u,v)_+ = B(u,\bar{v})$.

Let $(S_t,H_t)_{t\in(o,a)}$ be a L.S.O. on H_+ and suppose $\bar{H}_t\subset H_t$ $\forall t\in(o,a)$, we say that B is (S_t)-invariant and if $B(S_tu,\ \overline{S_tv}) =$

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