EXTRACTA MATHEMATICAE Vol. **27**, Núm. 2, 163 – 173 (2012)

On Dilation of Local Semigroups of Contractions and some Applications *

Ramón Bruzual, Marisela Domínguez, Ángel Padilla

Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela,
Apartado Postal 47686, *Caracas* 1041-A, *Venezuela*
ramonbruzual.ucv@gmail.com, ramon.bruzual@ciens.ucv.ve

Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela,
Apartado Postal 47159, *Caracas* 1041-A, *Venezuela*
marisela.dominguez@ciens.ucv.ve, dominguez.math@gmail.com

Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela
angel.padilla@ciens.ucv.ve

Presented by Alfonso Montes Received August 5, 2012

Abstract: We show that a multiplicative family of contractions on a separable Hilbert space, with parameter on the interval $[0, 1)$ of the dyadic rationals, has a unitary dilation with parameter on the dyadic rationals and values on a larger Hilbert space. This result is used to prove a dilation result for strongly continuous local semigroups of contractions. As an application we give results of extension of positive definite functions on the line, generalizing the Kreĭn extension theorem.

Key words: Operator semigroup, contraction operator, isometric operator, positive definite.

AMS *Subject Class.* (2010): 47D03, 43A35, 42A82

1. Basic notions and definitions

Results about dilation and extension of multiplicative families of contractions and local semigroups of contractions are a useful tool in some interpolation and dilation problems, see for example [1, 2, 3, 8]. Multiplicative families of operators, where the domain of the operators depends on the parameter, are considered in the papers [1, 2]. The techniques for obtaining the dilations and extensions are based on specials extensions of operators. In the papers [3, 8] the domain of the operators does not depend on the parameter and discretization techniques are used.

In this paper we use discretization techniques to extend, for a separable Hilbert space, some of the dilation results obtained in [1] and we give, as an application, some results about extension of positive definite functions.

*All authors were supported in part by the CDCH of the Univ. Central de Venezuela.

Let $(\Gamma, +)$ be an additive subgroup of the real numbers $(\mathbb{R}, +)$, let $a \in \mathbb{R}$, $a > 0$ or $a = +\infty$ and let $I_\Gamma = [0, a) \cap \Gamma$.

Definition 1. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space. A *multiplicative family of contractions* on $\mathcal{H}$ with parameter on I_Γ is a family $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$ such that:

(i) $\mathcal{H}(s)$ is a closed subspace of $\mathcal{H}$, $T(s) : \mathcal{H}(s) \to \mathcal{H}$ is a contraction operator, $\mathcal{H}(r) \subset \mathcal{H}(s)$ for $r, s \in I_\Gamma$, $s < r$ and $\mathcal{H}(0) = \mathcal{H}$, $T(0) = I_{\mathcal{H}}$.

(ii) If $r, s \in I_\Gamma$ are such that $r + s < a$ then $T(s)\mathcal{H}(r + s) \subset \mathcal{H}(r)$ and $T(r + s)h = T(r)T(s)h$ for all $h \in \mathcal{H}(r + s)$.

The multiplicative family is *strongly continuous* if for $r \in I_\Gamma$ and $f \in \mathcal{H}(r)$ the function $s \mapsto T(s)h$ is continuous on $[0, r] \cap I_\Gamma$.

If the operators $T(s)$ are isometries we will say that $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$ is a *multiplicative family of isometries*.

The following is an extension of a definition given in [1].

Definition 2. Let $\mathcal{H}$ be a Hilbert space. A *local semigroup of contractions* on $\mathcal{H}$ with parameter on I_Γ a multiplicative family of contractions $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$ such that

$$\bigcup_{r \in (x,a) \cap I_\Gamma} \mathcal{H}(r)$$

is dense in $\mathcal{H}(x)$ for all $x \in I_\Gamma$.

If the operators $T(s)$ are isometries we will say that $(T(s), \mathcal{H}(s))_{s \in I_\Gamma}$ is a *local semigroup of isometries*.

2. Multiplicative families of contractions with parameter on the diadic rational numbers

Let Δ be the set of the diadic rational numbers, that is

$$\Delta = \left\{ \frac{k}{2^n} \ : \ k \in \mathbb{Z}, \ n \in \mathbb{N} \right\}.$$

For each $m \in \mathbb{N}$ let

$$\Delta_m = \left\{ \frac{k}{2^m} \ : \ k \in \mathbb{Z} \right\},$$

$\Delta^+ = \{s \in \Delta \ : \ s \geq 0\}$ and $\Delta_m^+ = \{s \in \Delta_m \ : \ s \geq 0\}$.

Throughout this section $\mathcal{H}$ will be a separable Hilbert space, $I_\Delta = [0,1) \cap \Delta$ and $(T(s), \mathcal{H}(s))_{s \in I_\Delta}$ will denote a multiplicative family of contractions with parameter on I_Δ.

For $s \in I_\Delta$ let $V(s) : \mathcal{H} \to \mathcal{H}$ be the contraction operator defined by

$$V(s) = T(s) P^{\mathcal{H}}_{\mathcal{H}(s)},$$

where $P^{\mathcal{H}}_{\mathcal{H}(s)}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}(s)$.

Lemma 3. *If $m, k \in \mathbb{N}$ and $k < 2^m$ then*

$$\left(V\left(\tfrac{1}{2^m}\right)\right)^k \Big|_{\mathcal{H}\left(\frac{k}{2^m}\right)} = T\left(\tfrac{k}{2^m}\right).$$

Proof. We will use induction.

For $k = 2$.

Let $h \in \mathcal{H}\left(\frac{2}{2^m}\right)$. Then $h \in \mathcal{H}\left(\frac{1}{2^m}\right)$, so $P^{\mathcal{H}}_{\mathcal{H}\left(\frac{1}{2^m}\right)} h = h$. Since $\mathcal{H}\left(\frac{2}{2^m}\right) = \mathcal{H}\left(\frac{1}{2^m} + \frac{1}{2^m}\right)$ we have that $T\left(\frac{1}{2^m}\right) h \in \mathcal{H}\left(\frac{1}{2^m}\right)$, thus

$$P^{\mathcal{H}}_{\mathcal{H}\left(\frac{1}{2^m}\right)} T\left(\frac{1}{2^m}\right) h = T\left(\frac{1}{2^m}\right) h.$$

Therefore

$$\begin{aligned}
\left(V\left(\tfrac{1}{2^m}\right)\right)^2 h &= \left(T\left(\tfrac{1}{2^m}\right) P^{\mathcal{H}}_{\mathcal{H}\left(\frac{1}{2^m}\right)}\right) \left(T\left(\tfrac{1}{2^m}\right) P^{\mathcal{H}}_{\mathcal{H}\left(\frac{1}{2^m}\right)}\right) h \\
&= T\left(\tfrac{1}{2^m}\right) T\left(\tfrac{1}{2^m}\right) h \\
&= T\left(\tfrac{2}{2^m}\right) h
\end{aligned}$$

Suppose that $\left(V\left(\frac{1}{2^m}\right)\right)^k \Big|_{\mathcal{H}\left(\frac{k}{2^m}\right)} = T\left(\frac{k}{2^m}\right)$ and that $k+1 < 2^m$. Let $h \in \mathcal{H}\left(\frac{k+1}{2^m}\right)$, then $h \in \mathcal{H}\left(\frac{1}{2^m}\right)$ and $V\left(\frac{1}{2^m}\right) h = T\left(\frac{1}{2^m}\right) h$, so

$$\left(V\left(\tfrac{1}{2^m}\right)\right)^{k+1} h = \left(V\left(\tfrac{1}{2^m}\right)\right)^k V\left(\tfrac{1}{2^m}\right) h = \left(V\left(\tfrac{1}{2^m}\right)\right)^k T\left(\tfrac{1}{2^m}\right) h,$$

since $\mathcal{H}\left(\frac{k+1}{2^m}\right) = \mathcal{H}\left(\frac{1}{2^m} + \frac{k}{2^m}\right)$, we have that $T\left(\frac{1}{2^m}\right) h \in \mathcal{H}\left(\frac{k}{2^m}\right)$. From the induction hypothesis

$$\left(V\left(\tfrac{1}{2^m}\right)\right)^k T\left(\tfrac{1}{2^m}\right) h = T\left(\tfrac{k}{2^m}\right) T\left(\tfrac{1}{2^m}\right) h = T\left(\tfrac{k+1}{2^m}\right) h$$

thus

$$\left(V\left(\tfrac{1}{2^m}\right)\right)^{k+1} h = T\left(\tfrac{k+1}{2^m}\right) h.$$

∎

For $m \in \mathbb{N}$ we define $V^{(m)} = \big(V^{(m)}(s)\big)_{s\in\Delta_m^+}$ by

$$V^{(m)}\left(\tfrac{k}{2^m}\right) = \left(V_{\frac{1}{2^m}}\right)^k .$$

We have that $\big(V^{(m)}(s)\big)_{s\in\Delta_m^+}$ is a semigroup of contractions.

Proposition 4. *The function $F^{(m)} : \Delta_m \to L(\mathcal{H})$ defined by*

$$F^{(m)}(s) = \begin{cases} V^{(m)}(s) & \text{if } \mathrm{s} \geq 0, \\ \big(V^{(m)}(-s)\big)^* & \text{if } \mathrm{s} < 0, \end{cases}$$

is positive definite.

Proof. Considering the natural isomorphism between the group Δ_m and the integers group $\mathbb{Z}$ the result is obtained from the Neumark dilation theorem [7], see also Theorem 7.1 of Chapter 1 of [10]. ∎

The following result follows from a natural diagonalization procedure.

Proposition 5. *Let Υ be a topological space and let Λ be a numerable indices set. Let $\{X_\alpha(n)\}_{n=1}^\infty, \alpha \in \Lambda$ be a family of sequences in Υ such that for each $\alpha \in \Lambda$ every subsequence of $\{X_\alpha(n)\}_{n=1}^\infty$ has a convergent subsequence. Then there exists an increasing sequence $\{b(n)\}_{n=1}^\infty \subset \mathbb{N}$ such that the sequence $\{X_\alpha(b(n))\}_{n=1}^\infty$ converges for all $\alpha \in \Lambda$.*

Lemma 6. *There exists an increasing sequence of natural numbers $\{n_j\}_{j=1}^\infty$ such that for all $k, m \in \mathbb{N}$ and $h \in \mathcal{H}$ the sequence*

$$\left\{V^{(n_j)}\left(\tfrac{k}{2^m}\right) h\right\}_{j=1}^\infty$$

converges in the weak topology.

Proof. Since $\mathcal{H}$ is separable, there is a numerable set $\{h_i\}_{i=1}^{+\infty} \subset \mathcal{H}$ dense in $\mathcal{H}$.

Since the operators $V^{(n)}\left(\frac{k}{2^m}\right)$ are contractions, we have that for each $i, k, m \in \mathbb{N}$ the sequence $\left\{V^{(n)}\left(\frac{k}{2^m}\right) h_i\right\}_{n\geq m} \subset \mathcal{H}$ is bounded, so it contains a weakly convergent subsequence.

Applying Proposition 5 with the set $\Lambda = \{(i,k,m) : i,k,m \in \mathbb{N}\}$, $\Upsilon = \mathcal{H}$ with the weak topology and family

$$\left\{X_\alpha(n) = V^{(n)}\left(\tfrac{k}{2^m}\right) h_i ; \alpha = (i,k,m) \in \Lambda\right\}_{n\geq m}$$

we obtain that there is an increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that

$$\left\{V^{(n_j)}\left(\tfrac{k}{2^m}\right)h_i\right\}_{j=1}^{\infty}$$

converges weakly for all $i, k, m \in \mathbb{N}$.

We also have that $\left\{V^{(n_j)}\left(\tfrac{k}{2^m}\right)\right\}_{j=1}^{\infty} \subset L(\mathcal{H})$ is uniformly bounded and that $\{h_i\}_{i=1}^{\infty}$ is dense in $\mathcal{H}$, so for all $k, m \in \mathbb{N}$, the sequence

$$\left\{V^{(n_j)}\left(\tfrac{k}{2^m}\right)h\right\}_{j=1}^{\infty}$$

converges weakly for all $h \in \mathcal{H}$. ∎

Theorem 7. *Let $\mathcal{H}$ be a separable Hilbert space and let $(T(s), \mathcal{H}(s))_{s\in\Delta^+\cap[0,1)}$ be a multiplicative family of contractions on $\mathcal{H}$, then there exists a Hilbert space $\mathcal{G}$ containing $\mathcal{H}$ as a closed subspace and a group of unitary operators $(W(s))_{s\in\Delta} \subset L(\mathcal{G})$ such that*

$$T(s) = P_{\mathcal{H}}^{\mathcal{G}} W(s)|_{\mathcal{H}(s)}, \text{ for all } s \in \Delta^+ \cap [0,1).$$

Proof. For $s \in \Delta_+$ and $h \in \mathcal{H}$ we define $V^{(o)}(s)h$ as the weak limit of the sequence $\left\{V^{(n_j)}(s)h\right\}_{j=1}^{\infty}$ given in Lemma 6, so we have that $V^{(o)}(s) : \mathcal{H} \to \mathcal{H}$ is a linear contraction for all $s \in \Delta_+$ and

$$\left\langle V^{(o)}(s)\,h, g\right\rangle_{\mathcal{H}} = \lim_{j\longrightarrow\infty}\left\langle V^{(n_j)}(s)h, g\right\rangle_{\mathcal{H}}, \tag{2.1}$$

for all $s \in \Delta_+$ and $h, g \in \mathcal{H}$.

Let $F : \Delta \to L(\mathcal{H})$ defined by

$$F(s) = \begin{cases} V^{(o)}(s) & \text{if s} \geq 0, \\ \left(V_{-s}^{(o)}\right)^* & \text{if s} < 0. \end{cases}$$

From (2.1) we obtain that

$$\langle F(s)h, g\rangle_{\mathcal{H}} = \lim_{j\to\infty}\left\langle F^{(n_j)}(s)h, g\right\rangle_{\mathcal{H}}$$

for all $s \in \Delta$ and $h, g \in \mathcal{H}$, where $F^{(n_j)}$ s the function defined in the Proposition 4, which is positive definite. Since the weak limit of positive definite functions is positive definite, we have that F is positive definite.

From the Neumark dilation theorem it follows that there exists a Hilbert space $\mathcal{G}$ containing $\mathcal{H}$ as a closed subspace and a unitary group $(W(s))_{s\in\Delta} \subset L(\mathcal{G})$ such that

$$V^{(o)}(s) = P^{\mathcal{G}}_{\mathcal{H}} W(s)|_{\mathcal{H}}$$

for all $s \in \Delta^+$.

From Lemma 3 it follows that

$$V^{(o)}(s)|_{\mathcal{H}(s)} = T(s)$$

for all $s \in \Delta^+ \cap [0, 1)$.

Finally, we obtain that

$$T(s) = P^{\mathcal{G}}_{\mathcal{H}} W(s)|_{\mathcal{H}(s)}$$

for all $s \in \Delta^+ \cap [0, 1)$. ∎

Remark 8. This is a new result. It is important to no note that it was not necessary to assume the strong continuity of the multiplicative family of contractions.

Remark 9. Note that if, in Th. 7, we suppose that $(T(s), \mathcal{H}(s))_{s\in\Delta^+\cap\,[0,1)}$ is a multiplicative family of isometries on $\mathcal{H}$, then we obtain that

$$T(s) = W(s)|_{\mathcal{H}(s)}, \text{ for all } s \in \Delta^+ \cap [0, 1),$$

where $(W(s))_{s\in\Delta} \subset L(\mathcal{G})$ is a group of unitary operators.

3. Unitary dilation of strongly continuous local semigroups of contractions

Let $(\Gamma, +)$ be an additive subgroup of the real numbers such that $\Delta \subset \Gamma$. The following result is an extension, for the case of a separable Hilbert space, of Theorem 1 of [1].

Theorem 10. *Let $\mathcal{H}$ be a separable Hilbert space and let $(T(s), \mathcal{H}(s))_{s\in[0,1)\cap\Gamma}$ be a strongly continuous local semigroup of contractions on $\mathcal{H}$, then there exists a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace and a strongly continuous group of unitary operators $(U(s))_{s\in\mathbb{R}} \subset L(\mathcal{F})$ such that*

$$T(s) = P^{\mathcal{F}}_{\mathcal{H}} U(s)|_{\mathcal{H}(s)}, \text{ for all } s \in [0, 1) \cap \Gamma.$$

For the proof of the theorem we will need the following result.

Lemma 11. *Let $\mathcal{F}$ be a Hilbert space and let $(V(s))_{s\in\Delta} \subset L(\mathcal{F})$ be a strongly continuous group of unitary operators. Then there exists a strongly continuous group unitary operators $(U(s))_{s\in\mathbb{R}} \subset L(\mathcal{F})$ which extends $(V(s))_{s\in\Delta}$.*

Proof. We have that, for each $h \in \mathcal{F}$ the application $s \mapsto V(s)h$ is uniformly continuous on Δ.

Since Δ is dense in $\mathbb{R}$, for each $h \in \mathcal{F}$ there exists a unique continuous map $s \mapsto U(s)h$ from $\mathbb{R}$ in $\mathcal{F}$ such that if $t \in \Delta$ then

$$U(s)h = V(s)h \quad \text{for each } h \in \mathcal{F}.$$

If $s \in \mathbb{R}$ and $\{s_n\}_{n\geq 1} \subset \Delta$ is a sequence such that $\lim_{n\to\infty} s_n = s$ it holds that

$$U(s)h = \lim_{n\to\infty} V(s_n)h \text{ for all } h \in \mathcal{F}.$$

Finally, it is easy to check that $(U(s))_{s\in\mathbb{R}}$ is a strongly continuous group unitary operators. ∎

Proof of Theorem 10. From Theorem 7 it follows that there is a Hilbert space $\mathcal{G}$ containing $\mathcal{H}$ as a closed subspace and a group of unitary operators $(W(s))_{s\in\Delta} \subset L(\mathcal{G})$ such that

$$T(s) = P^{\mathcal{G}}_{\mathcal{H}} W(s)|_{\mathcal{H}(s)}, \text{ for all } s \in \Delta^+ \cap [0,1).$$

Let $\mathcal{F}$ be the closed subspace of $\mathcal{G}$ generated by $\{W(s)h \,:\, s \in \Delta,\, h \in \mathcal{H}\}$ and let $V(s)$ be the restriction of $W(s)$ to $\mathcal{F}$, then $(V(s))_{s\in\Delta} \subset L(\mathcal{F})$ is a unitary group such that

$$T(s) = P^{\mathcal{F}}_{\mathcal{H}} V(s)|_{\mathcal{H}(s)} \tag{3.1}$$

for all $s \in \Delta^+ \cap [0,1)$.

Now we will prove that $(V(s))_{s\in\Delta}$ is strongly continuous at 0 and therefore strongly continuous on Δ.

Since $\bigcup_{n=1}^{+\infty} \mathcal{H}\left(\frac{1}{2^n}\right)$ is dense on $\mathcal{H}$ we have that the set

$$\left\{ V(s)h \,:\, s \in \Delta,\, h \in \bigcup_{n=1}^{+\infty} \mathcal{H}\left(\tfrac{1}{2^n}\right) \right\}$$

is dense on $\mathcal{F}$.

Therefore it is enough to show that

$$\lim_{\substack{s\to 0\\ s\in\Delta}} \|V(s)h-h\|_{\mathcal{H}} = 0, \qquad \text{for all } h\in\bigcup_{n=1}^{+\infty}\mathcal{H}\left(\tfrac{1}{2^n}\right).$$

Let $h\in\bigcup_{n=1}^{+\infty}\mathcal{H}\left(\frac{1}{2^n}\right)$, then there exists a natural number n_o such that $h\in\mathcal{H}\left(\frac{1}{2^{n_o}}\right)$.

Let $s\in\Delta^+\cap[0,1)$ such that $s<\frac{1}{2^{n_o}}$, then $h\in\mathcal{H}(s)$ and we have that

$$\begin{aligned}
\|V(s)h-h\|^2_{\mathcal{H}} &= \langle V(s)h-h, V(s)h-h\rangle_{\mathcal{H}}\\
&= \langle V(s)h, V(s)h\rangle_{\mathcal{H}} - 2Re\left(\langle V(s)h,h\rangle_{\mathcal{H}}\right) + \langle h,h\rangle_{\mathcal{H}}\\
&= 2\langle h,h\rangle_{\mathcal{H}} - 2Re\left(\langle P^{\mathcal{H}}_{\mathcal{H}} V(s)h,h\rangle_{\mathcal{H}}\right)\\
&= 2\langle h,h\rangle_{\mathcal{H}} - 2Re\left(\langle T(s)h,h\rangle_{\mathcal{H}}\right).
\end{aligned}$$

So the result follows from the strong continuity of $(T(s),\mathcal{H}(s))_{s\in[0,1)\cap\Gamma}$.

From Lemma 11 it follows that there exists a strongly continuous group of unitary operators $(U(s))_{s\in\mathbb{R}}\subset L(\mathcal{F})$ such that

$$U(s)=V(s) \tag{3.2}$$

for all $s\in\Delta$.

Let us show that $T(s)=P^{\mathcal{F}}_{\mathcal{H}}U(s)|_{\mathcal{H}(s)}$ for all $s\in[0,1)\cap\Gamma$.

Let $s\in[0,1)\cap\Gamma$ and $h\in\mathcal{H}(s)$. Since Δ is dense on $\mathbb{R}$ there exists an increasing sequence $\{s_n\}_{n\geq 1}$ in $\Delta^+\cap[0,1)$ such that $\lim_{n\to\infty}s_n=s$. We have that $\mathcal{H}(s)\subset\mathcal{H}(s_n)$ for all n.

From the strong continuity of the local semigroup and the group $(U(s))$ we have

$$\begin{aligned}
T(s)h &= \lim_{n\longrightarrow\infty} T(s_n)h = \lim_{n\longrightarrow\infty} P^{\mathcal{F}}_{\mathcal{H}}V(s_n)h\\
&= \lim_{n\longrightarrow\infty} P^{\mathcal{F}}_{\mathcal{H}}U(s_n)h\\
&= P^{\mathcal{F}}_{\mathcal{H}}U(s)h.
\end{aligned}$$

∎

Remark 12. Note that if $(T(s),\mathcal{H}(s))_{s\in\Delta^+\cap[0,1)}$ is a local semigroup of isometries on $\mathcal{H}$, then

$$T(s)=U(s)|_{\mathcal{H}(s)}, \text{ for all } s\in\Delta^+\cap[0,1).$$

Remark 13. If $a\,\Delta\subset\Gamma$, with a natural change of variables this theorem can be extended for an interval of the form $[0,a)$ for $a\in\mathbb{R}$ and $a>0$, instead of $[0,1)$.

4. Extension of positive definite functions

As in Section 1, let $(\Gamma, +)$ be an additive subgroup of the real numbers $(\mathbb{R}, +)$.

Let $a \in \mathbb{R}$ such that $a > 0$ and let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}})$ be a Hilbert space, recall that a function $f : (-a, a) \cap \Gamma \to L(\mathcal{E})$ is said to be *positive definite* if

$$\sum_{x,y\in[0,a)\cap\Gamma} \langle f(x-y)h(x), h(y)\rangle_{\mathcal{E}} \geq 0$$

for all functions $h : [0, a) \cap \Gamma \to \mathcal{E}$ of finite support. We will suppose that $f(0) = I_{\mathcal{E}}$, the identity operator.

M. G. Kreĭn [6] proved that a continuous scalar valued positive definite function, on an interval of the real line, can be extended to a continuous positive definite function on the whole line. This result was extended for strongly continuous operator valued functions by M. L. Gorbachuck [4].

A multiplicative family of isometric operators can be associated, in a natural way, to a positive definite function, see for example [1, 2, 5, 9] for more details. This correspondence is established as follows.

Let

$$\mathcal{D} = \{h : [0, a) \cap \Gamma \to \mathcal{E} \mid \text{ support of } h \text{ is finite }\}.$$

Then $\mathcal{D}$ is a linear space, for $h, h' \in \mathcal{D}$ we define

$$\langle h, h'\rangle_{\mathcal{D}} = \sum_{x,y\in[0,a)\cap\Gamma} \langle f(x-y)h(x), h'(y)\rangle_{\mathcal{E}},$$

then $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ is a, possibly degenerate, positive sesquilinear form on $\mathcal{D}$. Let $\mathcal{H}$ be the completion of $\mathcal{D}$, after the natural quotient.

For $r \in [0, a) \cap \Gamma$ we define

$$\mathcal{D}(r) = \{h \in \mathcal{D} \mid \text{ support}(h) \subset [0, a-r) \cap \Gamma\}$$

and $S(r) : \mathcal{D}(r) \to \mathcal{D}$ by

$$S(r)h(x) = \begin{cases} h(x-r) & \text{if } x \in [r, a), \\ 0 & \text{if } x \in [0, r). \end{cases}$$

It holds that $S(r)$ is an isometric operator and, if $\mathcal{H}(r)$ is the closure of $\mathcal{D}(r)$ in $\mathcal{H}$, then $S(r)$ can be extended to an isometric operator from $\mathcal{H}(r)$ to $\mathcal{H}$, denoting this extension by $T(r)$, we have that $(T(r), \mathcal{H}(r))_{r\in[0,a)\cap\Gamma}$ is a multiplicative family of contractions. It is easy to check that, if f is strongly continuous, then $(T(r), \mathcal{H}(r))_{r\in[0,a)\cap\Gamma}$ is a strongly continuous local semigroup of isometries.

Since $f(0) = I_{\mathcal{E}}$ there exists a natural immersion of $\mathcal{E}$ into $\mathcal{H}$ and it holds that

$$\langle f(x)h, h'\rangle_{\mathcal{E}} = \langle T(x)h, h'\rangle_{\mathcal{H}}$$

for $h, h' \in \mathcal{E}$ and $x \in [0, a) \cap \Gamma$. If the multiplicative family of isometries $(T(r), \mathcal{H}(r))_{r\in[0,a)\cap\Gamma}$ could be extended to an unitary group $(U(r))_{r\in\mathbb{R}}$ on a larger Hilbert space $\mathcal{F}$ we would have

$$f(x) = P^{\mathcal{F}}_{\mathcal{E}} U(x)|_{\mathcal{E}} \quad \text{for } x \in (-a, a) \cap \Gamma.$$

So the extension to an unitary group of the family $(T(r), \mathcal{H}(r))_{r\in[0,a)\cap\Gamma}$ is a sufficient condition for the extension of f to a positive definite function in the whole line. If the space $\mathcal{E}$ is separable, then the corresponding space $\mathcal{H}$ is also separable. Therefore from Theorems 7 and 10 we obtain the following two results.

THEOREM 14. *Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}})$ be a separable Hilbert space and let $f : \Delta \cap (-1, 1) \to \mathcal{E}$ be a positive definite function such that $f(0) = I_{\mathcal{E}}$, then f can be extended to a positive definite function $F : \Delta \to \mathcal{E}$.*

THEOREM 15. *Suppose that $a \in [0, +\infty)$ an that $a\,\Delta \subset \Gamma$. Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}})$ be a separable Hilbert space and let $f : \Gamma \cap (-a, a) \to \mathcal{E}$ be a strongly continuous positive definite function such that $f(0) = I_{\mathcal{E}}$, then f can be extended to a strongly continuous positive definite function $F : \mathbb{R} \to \mathcal{E}$.*

Remark 16. Note that it was not necessary to assume the strong continuity of the function in Theorem 14.

References

[1] R. Bruzual, Local semigroups of contrations and some applications to Fourier representation theorems, *Integr. Equ. Oper. Theory* **10** (1987), 780 – 801.

[2] R. Bruzual, M. Domínguez, Extensions of operator valued positive definite functions and commutant lifting on ordered groups, *J. Funct. Anal.* **185** (2) (2001), 456 – 473.

[3] R. Bruzual, M. Domínguez, M. Montilla, On unitary dilations of two-parameter semigroups of contractions and continuous commutant lifting, *Acta Sci. Math.* **77** (3-4) (2011), 607 – 620.

[4] M.L. Gorbachuck, Representation of positive definite operator functions, *Ukrainian Math. J.* **17** (1965), 29 – 46.

[5] M. Grossman, H. Langer, Über indexerhaltende Erweiterungen eines hermiteschen operators in Pontrjaginraum, *Math. Nachr.* **64** (1974), 289 – 317.

[6] M.G. Kreĭn , Sur le problème du prolongement des fonctions hermitiennes positives et continues, *Dokl. Akad. Nauk. SSSR* **26** (1940), 17 – 22.

[7] M.A. Neumark, Spectral functions of a symmetric operator, *Bull. Acad. Sci. URSS, Sér. Math.* **4** (1940), 277 – 318.

[8] M. Ptak, Unitary dilations of multiparameter semigroups of operators, *Ann. Polon. Math.* **45** (1985), 237 – 243.

[9] Z. Sasvári, "Positive Definite and Definitizable Functions", Akademie Verlag, Berlin, 1994.

[10] B. Sz.-Nagy, C. Foias, "Harmonic Analysis of Operators on Hilbert Space", North Holland Publishing Co., Amsterdam-London, 1970.